May 4, 2016

M001: Mathematics

Homework 6-Solution

Problem 1

Find the partial derivatives $\partial f \,/\, \partial x$ and $\partial f \,/\, \partial y$ of the functions

$f(x,y) = x^5 + 3x^3y^2 + 3xy^4$ $\quad$ $f(x,y) = \ln\left(x + \sqrt{x^2+y^2}\right)$ $\quad$ $f(x,y) = -xye^{-x^2-y^2}$

$f(x,y) = \dfrac{x-y}{x+y}$ $\quad$ $f(x,y) = \dfrac{-3y}{x^2+y^2+1}$

Solution:

$$\begin{aligned} f(x,y) &= x^5 + 3x^3y^2 + 3xy^4 \\ \frac{\partial f}{\partial x} &= 5x^4 + 9x^2y^2 + 3y^4 \\ \frac{\partial f}{\partial y} &= 6x^3y + 12xy^3 \end{aligned}$$

$$\begin{aligned} f(x,y) &= \ln\left(x + \sqrt{x^2+y^2}\right) \\ \frac{\partial f}{\partial x} &= \frac{1}{x+\sqrt{x^2+y^2}}\left(1 + \frac{2x}{2\sqrt{x^2+y^2}}\right) \\ \frac{\partial f}{\partial y} &= \frac{1}{x+\sqrt{x^2+y^2}}\frac{2y}{2\sqrt{x^2+y^2}} \end{aligned}$$

$$\begin{aligned} f(x,y) &= -xye^{-x^2-y^2} \\ \frac{\partial f}{\partial x} &= -y\left(1-2x^2\right)e^{-x^2-y^2} \\ \frac{\partial f}{\partial y} &= -x\left(1-2y^2\right)e^{-x^2-y^2} \end{aligned}$$

$$\begin{aligned} f(x,y) &= \frac{x-y}{x+y} \\ \frac{\partial f}{\partial x} &= \frac{\left(x+y\right)-\left(x-y\right)}{\left(x+y\right)^2} &&= \frac{2y}{\left(x+y\right)^2} \\ \frac{\partial f}{\partial y} &= \frac{-\left(x+y\right)-\left(x-y\right)}{\left(x+y\right)^2} &&= \frac{-2x}{\left(x+y\right)^2} \end{aligned}$$

$$\begin{aligned} f(x,y) &= \frac{-3y}{x^2+y^2+1} \\ \frac{\partial f}{\partial x} &= \frac{3y2x}{\left(x^2+y^2+1\right)^2} &&= \frac{6xy}{\left(x^2+y^2+1\right)^2} \\ \frac{\partial f}{\partial y} &= \frac{-3\left(x^2+y^2+1\right)+3y2y}{\left(x^2+y^2+1\right)^2} &&= \frac{-3\left(x^2+y^2+1\right)+6y^2}{\left(x^2+y^2+1\right)^2} \end{aligned}$$

Problem 2
Find the maximum rate of change of *f* at the given point and the direction in which it occurs.

$$f(x,y) = xe^{-y} + 3y \qquad (1,\ 0)$$

$$f(x,y) = \ln\left(x^2 + y^2\right) \qquad (1,\ 2)$$

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} \qquad (4,2,1)$$

$$f(x,y) = xe^{-y} + 3y \qquad (1,\ 0)$$

$$f_x = e^{-y} \qquad \rightarrow f_x(1,0) = e^0 = 1$$

$$f_y \quad -xe^{-y} + 3 \qquad \rightarrow f_y(1,0) = -1 \cdot e^0 + 3 = 2$$

$$\text{grad } f = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \left|\text{grad } f\right| = \sqrt{1^2 + 2^2} = \sqrt{5} = 2.236$$

$$f(x,y) = \ln\left(x^2 + y^2\right) \qquad (1,\ 2)$$

$$f_x = \frac{2x}{x^2 + y^2} \qquad \rightarrow f_x(1,2) = \frac{2 \cdot 1}{1^2 + 2^2} = \frac{2}{5}$$

$$f_y = \frac{2y}{x^2 + y^2} \qquad \rightarrow f_y(1,2) = \frac{2 \cdot 2}{1^2 + 2^2} = \frac{4}{5}$$

$$\text{grad } f = \frac{1}{5}\begin{pmatrix} 2 \\ 4 \end{pmatrix} \qquad \left|\text{grad } f\right| = \frac{1}{5}\sqrt{2^2 + 4^2} = \frac{1}{5}\sqrt{20} = 0.8944$$

$$f(x,y,z) = \frac{x}{y} + \frac{y}{z} \qquad (4,2,1)$$

$$f_x = \frac{1}{y} \qquad \rightarrow f_x(4,2,1) = \frac{1}{2}$$

$$f_y = -\frac{x}{y^2} + \frac{1}{z} \qquad \rightarrow f_y(4,2,1) = -\frac{4}{2^2} + 1 = 0$$

$$f_z = -\frac{y}{z^2} \qquad \rightarrow f_y(4,2,1) = -\frac{2}{1^2} = -2$$

$$\text{grad } f = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix} \qquad \left|\text{grad } f\right| = \frac{1}{2}\sqrt{1^2 + 0^2 + \left(-4\right)^2} = \frac{1}{2}\sqrt{17} = 2.062$$

Problem 3

Suppose that the temperature at a point in space is given by

$$T(x,y,z) = \frac{80}{1+x^2+2y^2+3z^2}$$

where T is measured in degrees Celsius and x,y,z in meters. In which direction does the temperature increase fastest at the point (1,1-2)? What is the maximum rate of increase?

Solution

$$T_x = \frac{-160x}{\left(1+x^2+2y^2+3z^2\right)^2} \rightarrow T_x(1,1,-2) = \frac{-160\cdot 1}{\left[1+1^2+2\cdot 1^2+3\cdot\left(-2\right)^2\right]^2} = -\frac{5}{8}$$

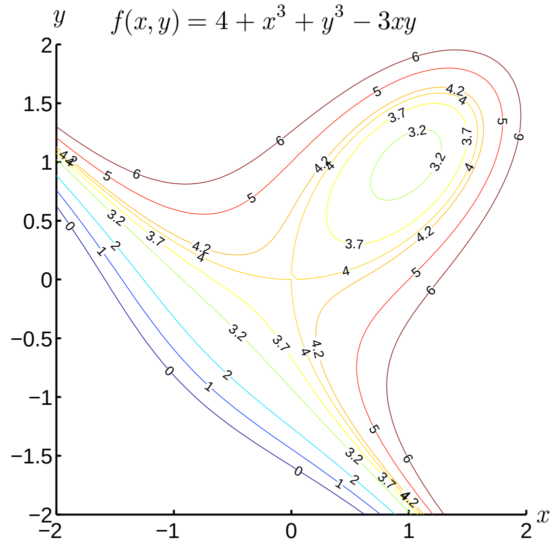
$$T_y = \frac{-320y}{\left(1+x^2+2y^2+3z^2\right)^2} \rightarrow T_y(1,1,-2) = \frac{-320\cdot 1}{\left[1+1^2+2\cdot 1^2+3\cdot\left(-2\right)^2\right]^2} = -\frac{5}{4}$$

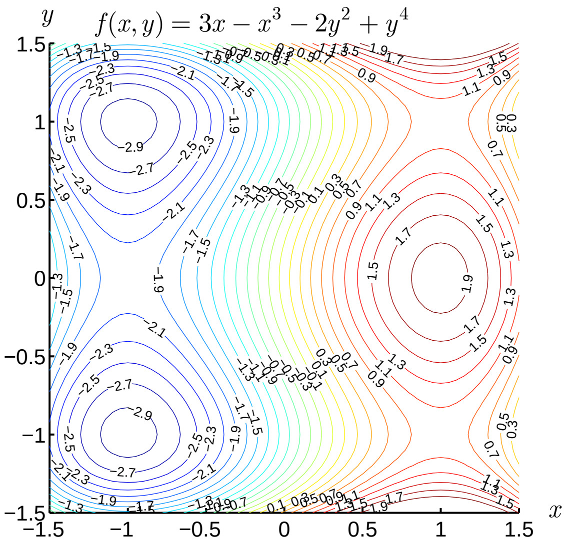
$$T_z = \frac{-480z}{\left(1+x^2+2y^2+3z^2\right)^2} \rightarrow T_z(1,1,-2) = \frac{480\cdot 2}{\left[1+1^2+2\cdot 1^2+3\cdot\left(-2\right)^2\right]^2} = \frac{30}{8}$$

$$\operatorname{grad} T = \frac{1}{8}\begin{pmatrix}-5\\-10\\30\end{pmatrix} \rightarrow \left|\operatorname{grad} T\right| = \frac{1}{8}\sqrt{5^2+10^2+30^2} = \frac{1}{8}\sqrt{1025} = \frac{5}{8}\sqrt{41} = 4.002\,\frac{\circ}{\mathrm{m}}$$

Problem 4

Use the level curves in the figure to predict the location of the critical points of f and whether f has a saddle point or local minimum or maximum at each of those points. Then analyze to confirm your predictions.





(a)

$f_x = 3x^2 - 3y = 0 \rightarrow y = x^2$

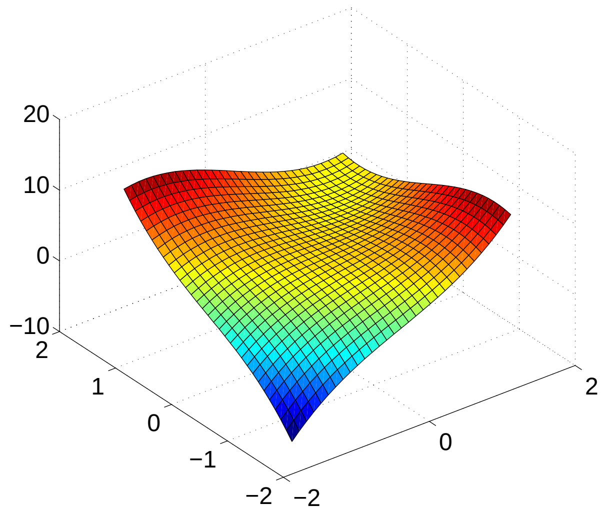
$f_y = 3y^2 - 3x = 0 \rightarrow y = \pm\sqrt{x}$

$\rightarrow x = 0, y = 0 \vee x = 1, y = 1$

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = -3$



x	y	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	**Type**
0	0	0	0	-3	-9	Saddle point
1	1	6	6	-3	27	Minimum

(b)

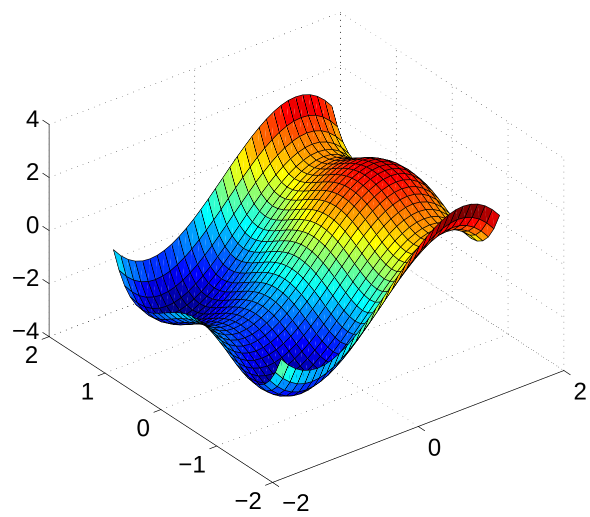
$f_x = 3 - 3x^2 = 0 \rightarrow x = \pm 1$

$f_y = -4y + 4y^3 = 0 \rightarrow y = \pm\sqrt{1} \vee y = 0$

$f_{xx} = -6x$

$f_{yy} = -4 + 12y^2$

$f_{xy} = 0$



x	y	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	**Type**
-1	-1	6	8	0	48	Minimum
-1	0	6	-4	0	-24	Saddle point
-1	1	6	8	0	48	Minimum
1	-1	-6	8	0	-48	Saddle point
1	0	-6	-4	0	24	Maximum
1	1	-6	8	0	-48	Saddle point

Problem 5
Find the shortest distance from the point (1,0,-2) to the plane $x+2y+z=4$. Hint: Start by writing

$$d=\sqrt{\left(x-1\right)^2+y^2+\left(z+2\right)^2}$$

for the distance d from any point x,y,z to the point (1,0,-2). Then solve the equation of the plane for z and substitute into the expression for d . Note that we can minimize d by minimizing the simpler expression for d^2 .

Solution:

$$\begin{aligned}
z &= 4-x-2y \\
d^2 &= f(x,y)=\left(x-1\right)^2+y^2+\left(z+2\right)^2 \\
&= \left(x-1\right)^2+y^2+\left(6-x-2y\right)^2 \\
f_x &= 2\left(x-1\right)-2\left(6-x-2y\right) \\
&= 4x+4y-14 \\
f_y &= 2y-4\left(6-x-2y\right) \\
&= 4x+10y-24
\end{aligned}$$

$$f_x = 0 \rightarrow x=\frac{1}{4}\left(14-4y\right)$$

$$f_y(x=\frac{1}{4}\left(14-4y\right),y)=4\cdot\frac{1}{4}\left(14-4y\right)+10y-24=6y-10=0$$

$$\rightarrow y=\frac{5}{3}$$

$$\rightarrow x=\frac{1}{4}\left(14-4\cdot\frac{5}{3}\right)=\frac{11}{6}$$

$$\begin{aligned}
d &= \sqrt{\left(x-1\right)^2+y^2+\left(6-x-2y\right)^2} \\
&= \sqrt{\left(\frac{11}{6}-1\right)^2+\left(\frac{5}{3}\right)^2+\left(6-\frac{11}{6}-2\cdot\frac{5}{3}\right)^2} \\
&= \sqrt{\frac{25}{36}+\frac{25}{9}+\frac{25}{36}} \\
&= \sqrt{\frac{150}{36}}=\sqrt{\frac{6\cdot 25}{36}} \\
&= \frac{5}{6}\sqrt{6}
\end{aligned}$$

Problem 6

Find and classify the critical points of the function

$$f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Solution:

$$f_x = 20xy - 10x - 4x^3 = x\left(20y - 10 - 4x^2\right) = 0 \rightarrow x = 0 \ \vee x = \pm\sqrt{\frac{20y-10}{4}}$$

$$f_y = 10x^2 - 8y - 8y^3$$

$$x = 0 \rightarrow \frac{\partial f}{\partial y} = -8y - 8y^3 = 0 \rightarrow y = 0 \vee 8y^2 + 8 = 0$$

$$\rightarrow \text{ for } x = 0, \text{only } y = 0 \,\text{possible (since } 8y^2 + 8 \text{ cannot be zero)}$$

$$x = \pm\sqrt{\frac{20y-10}{4}} \rightarrow \frac{\partial f}{\partial y} = 10\cdot\left(\frac{20y-10}{4}\right) - 8y - 8y^3 = 50y - 25 - 8y - 8y^3 = 0 \rightarrow 8y^3 - 42y + 25 = 0$$

solve electronically $\rightarrow y_1 = -2.545, y_2 = 1.898, y_3 = 0.6468$

$$x_1 = \pm\sqrt{\frac{20\cdot(-2.545)-10}{4}} \quad \rightarrow \text{ complex number (no solution)}$$

$$x_2 = \pm\sqrt{\frac{20\cdot 1.898 - 10}{4}} \quad = \pm 2.644$$

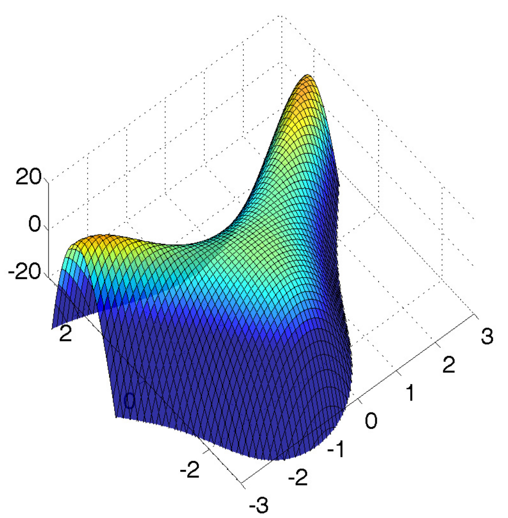
$$x_3 = \pm\sqrt{\frac{20\cdot 0.6468 - 10}{4}} \quad = \pm 0.857$$

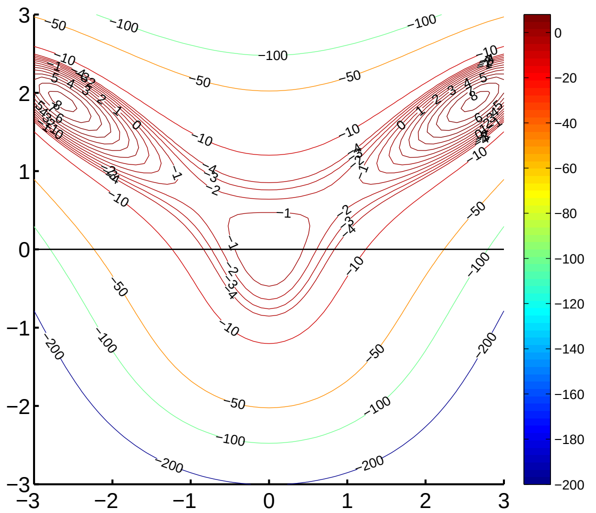
$$f_{xx} = 20y - 10 - 12x^2 = x\left(20y - 10 - 4x^2\right) = 0 \rightarrow x = 0 \ \vee x = \pm\sqrt{\frac{20y-10}{4}}$$

$$f_{yy} = -8 - 24y^2$$

$$f_{xy} = 20x$$

x	y	f_{xx}	f_{yy}	f_{xy}	$f_{xx}f_{yy} - f_{xy}^2$	**Type**
0	0	-10.0000	-8.0000	0	80	Maximum
-2.6439	1.8980	-55.9200	-94.4577	-52.8772	2486.1	Maximum
2.6439	1.8980	-55.9200	-94.4577	52.8772	2486.1	Maximum
-0.8567	0.6468	-5.8720	-18.0404	-17.1348	-187.7	Saddle point
0.8567	0.6468	-5.8720	-18.0404	17.1348	-187.7	Saddle point





Problem 7
Use the Newton-Raphson method to solve the nonlinear system of equations

$$2x_1^2 + 4x_1x_2 + 7x_2^2 + 3x_1 + 4x_2 = 77.75$$
$$x_1^2 + 6x_1x_2 + 3x_2^2 + 5x_1 + 2x_2 = 56.00$$

Solution:
Solve

$$f_1(x_1,x_2) = 2x_1^2 + 4x_1x_2 + 7x_2^2 + 3x_1 + 4x_2 - 77.75 = 0$$
$$f_2(x_1,x_2) = x_1^2 + 6x_1x_2 + 3x_2^2 + 5x_1 + 2x_2 - 56.00 = 0$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 + 4x_2 + 3 & 4x_1 + 14x_2 + 4 \\ 2x_1 + 6x_2 + 5 & 6x_1 + 6x_2 + 2 \end{bmatrix}$$

i	1	2	3	4
$\mathbf{x}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 4.8929 \\ 15.7679 \end{bmatrix}$	$\begin{bmatrix} 2.5684 \\ 8.0003 \end{bmatrix}$	$\begin{bmatrix} 1.6183 \\ 4.3389 \end{bmatrix}$
$\mathbf{f}$	$\begin{bmatrix} -77.75 \\ -56.00 \end{bmatrix}$	$\begin{bmatrix} 2096.9 \\ 1232.7 \end{bmatrix}$	$\begin{bmatrix} 505.4 \\ 294.7 \end{bmatrix}$	etc.
$\mathbf{J}$	$\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$	$\begin{bmatrix} 85.6429 & 244.3214 \\ 109.3929 & 125.9643 \end{bmatrix}$	$\begin{bmatrix} 45.2748 & 126.2776 \\ 58.1385 & 65.4121 \end{bmatrix}$	etc.

The solution converges to $x_1 = 1.5000,\ x_2 = 2.5000$.

Problem 8

Derive the location four Gauss points and the corresponding weights, requiring that the numerical integration gives the exact results for any seven-degree polynomial (eight coefficients). You will have to write a nonlinear system of equations with eight unknowns. Use a program other than EXCEL. The function and main program below should help you solve the problem.

```
function [y,yp]= nonlinearmultivariatefunctions(i,x)
if i==1
%derive four Gauss points and corresponding weights
%such that cubic polynomial is integrated exactly
%x(1), x(2) are sample points
%x(3), x(4) are weights
    y(1,1)  = x(3) + x(4)-2;
    y(2,1)  = x(1)*x(3) + x(2)*x(4);
    y(3,1)  = x(1)^2*x(3) + x(2)^2*x(4)-2/3;
    y(4,1)  = x(1)^3*x(3) + x(2)^3*x(4);

    yp = zeros(4,4);
    yp(1,3) = 1
    yp(1,4) = 1
    yp(2,:) = [x(3)            x(4)            x(1)       x(2) ];
    yp(3,:) = [2*x(1)  *x(3)   2*x(2)  *x(4)   x(1)^2     x(2)^2 ];
    yp(4,:) = [3*x(1)^2*x(3)   3*x(2)^2*x(4)   x(1)^3     x(2)^3 ];
end

if i==2
    %other nonlinear systems of equations
end

%starting point
x(:,1)=[-1 1 1 1]'
tol = 1
i   = 1
while tol > 0.000001
    [y(:,i),yp] = nonlinearmultivariatefunctions(1,x(:,i));
    delx(:,i+1)  = -inv(yp) * y(:,i);
    x(:,i+1)     =  x(:,i) + delx(:,i+1)
    tol          =  norm( delx(:,i+1) );
    i            =  i + 1;
end
```

Conditions for GAUSS points (see lecture notes)

$$\begin{aligned} w_1 + w_2 &= 2 \\ \xi_1 w_1 + \xi_2 w_2 &= 0 \\ \xi_1^2 w_1 + \xi_2^2 w_2 &= \frac{2}{3} \\ \xi_1^3 w_1 + \xi_2^3 w_2 &= 0 \end{aligned}$$

Thus, solve

$$\begin{aligned} f_1(x_1,x_2,x_3,x_4) = x_3 + x_4 - 2 &= 0 \\ f_2(x_1,x_2,x_3,x_4) = x_1 x_3 + x_2 x_4 &= 0 \\ f_3(x_1,x_2,x_3,x_4) = x_1^2 x_3 + x_2^2 x_4 - \frac{2}{3} &= 0 \\ f_4(x_1,x_2,x_3,x_4) = x_1^3 x_3 + x_2^3 x_4 &= 0 \end{aligned}$$

Jacobian Matrix

$$\frac{\partial f_1}{\partial x_1} = 0 \qquad \frac{\partial f_1}{\partial x_2} = 0 \qquad \frac{\partial f_1}{\partial x_3} = 1 \qquad \frac{\partial f_1}{\partial x_4} = 1$$

$$\frac{\partial f_2}{\partial x_1} = x_3 \qquad \frac{\partial f_2}{\partial x_2} = x_4 \qquad \frac{\partial f_2}{\partial x_3} = x_1 \qquad \frac{\partial f_2}{\partial x_4} = x_2$$

$$\frac{\partial f_3}{\partial x_1} = 2x_1 x_3 \qquad \frac{\partial f_3}{\partial x_2} = 2x_2 x_4 \qquad \frac{\partial f_3}{\partial x_3} = x_1^2 \qquad \frac{\partial f_3}{\partial x_4} = x_2^2$$

$$\frac{\partial f_4}{\partial x_1} = 3x_1^2 x_3 \qquad \frac{\partial f_4}{\partial x_2} = 3x_2^2 x_4 \qquad \frac{\partial f_4}{\partial x_3} = x_1^3 \qquad \frac{\partial f_4}{\partial x_4} = x_2^3$$

Iteration not shown.